

## Exam

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

### Problem 1 : (120 pt)

Let

$$A = \begin{pmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{pmatrix}$$

1. Why  $A$  is orthogonally diagonalizable ?
2. Give the definition of characteristic polynomial of  $A$  AND compute it.
3. Prove that 1 is a root of this characteristic polynomial. THEN, factorize  $(\lambda - 1)$  from the characteristic polynomial  $p(\lambda)$ , that is, find integers  $a$ ,  $b$  and  $c$  such that

$$p(\lambda) = (\lambda - 1)(a\lambda^2 + b\lambda + c)$$

4. Deduce all the eigenvalues for  $A$  and their multiplicities.
5. Without any computations, deduce the dimension of each eigenspace. Explain.
6. Compute an orthonormal basis for each eigenspace.
7. Orthogonally diagonalize  $A$ . Compute the inverse of  $P$  in the formula,  $P$  being the matrix which orthogonally diagonalize  $A$ .
8. Deduce the spectral decomposition of  $A$ .
9. Compute  $A^n$ , for  $n \in \mathbb{N}$ , (in term of  $n$ ).
10. We define the quadratic form  $Q(x) = x^T A x$  with  $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ . Find a change of variables  $y = Ux$  such that the new quadratic form induce by this change of variables has no cross product.
11. Is  $Q$  indefinite ? positive definite ? negative definite ? Justify.

12. Let  $x_0 \in \mathbb{R}^3$ . Consider the differential equation

$$Ax_k = x_{k+1}$$

Give the form of the general solution for this equation in term of the eigenvectors and the eigenvalues of  $A$ .

Problem 2 : (70 pt)

Let

$$A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix}$$

1. Let  $A = U\Sigma V^T$  be the singular value decomposition of  $A$ . Describe and give the properties that  $U$ ,  $\Sigma$  and  $V$ .
2. Give the definition of the singular value of  $A$  AND find them. Deduce the rank of  $A$  AND  $\Sigma$ .
3. Find the columns of  $V$ . Explain all your work.
4. Deduce a basis for  $\text{Col}(A)$ , AND deduce the first column  $u_1$  of  $U$ .
5. Write the equation satisfied by any vector  $x \in \mathbb{R}^3$  orthogonal to  $u_1$  AND find an orthonormal basis for the solution set of this equation.
6. Deduce  $U$ .
7. Give the singular value decomposition of  $A$ .

Problem 3 : (20 pt)

Given subspaces  $H$  and  $K$  of a vector space  $V$ , the sum of  $H$  and  $K$ , written as  $H + K$  is the set of all vectors in  $V$  that can be written as the sum of two vectors one in  $H$  and the other in  $K$ , that is

$$H + K = \{w, w = u + v, \text{ for some } u \text{ in } H \text{ and some } v \text{ in } K\}$$

1. Show that  $H + K$  is a subspace of  $V$ .
2. Suppose that  $H$  and  $K$  are finite dimensional with basis respectively  $\{h_1, \dots, h_s\}$  and  $\{k_1, \dots, k_t\}$ . Prove that  $\{h_1, \dots, h_s, k_1, \dots, k_t\}$  is a spanning set for  $H + K$ .

Problem 4 : (10 pt)

Let  $T : V \rightarrow V$  and  $S : V \rightarrow V$  be linear maps where  $V$  is a vector space. Prove that the composite  $T \circ S$  is also a linear map.

Problem 5 : (20 pt)

1. Prove that the set  $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$  is a basis for  $\mathcal{P}_2$  the set of polynomial of degree at most 2;
2. Find the coordinate of  $p(t) = 1 + 3t - 6t^2$  relative to  $\mathcal{B}$ .

**Problem 6 : (20 pt)**

Let  $V$  be the space  $C[0,1]$  of the continuous function on  $[0,1]$ . We define on it the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt, \forall f, g \in C([0,1])$$

Let  $W$  be the subspace of  $V$  generated by  $1, t, t^2$ .

1. Identify  $W$  with a vector space seen in class.
2. Construct a orthonormal basis for  $W$  for the inner product defined above, using the Gram Schmidt process.